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# THREE-DIMENSIONAL KIRSCH PROBLEM FOR A TRANSTROPIC PLATE*

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Qualitative and quantitative study of the concentration of stresses in the Kirsch problem for isotropic plates is carried out in the three-dimensional formulation using the superposition method /1/ and the method of homogeneous solutions /2/. An asymptotic method of solving the Kirsch problem for transtropic plates is given in /3/. Below the problem in question is solved for the transtropic bodies of finite dimensions.

Let $V = \{1 \leqslant r < \infty,\ 0 \leqslant \theta \leqslant 2\pi,\ |\zeta| \leqslant 1\}$ be a region occupied by a plate made of transtropic material. The material is characterized by the elastic parameters $\nu, \nu_z, \nu_2 = \nu_z E / E_z$, $s_0^2 = G/G_z$ ($\nu_z$, $E_z$, $G_z$ denoting the Poisson's ratio, Young's modulus and shear modulus in the planes perpendicular to the isotropy planes). We assume that the end faces $S_\pm$ of the plate and its side surface $\Omega$ are free of external loads and that at infinity the plate is under uniaxial tension, i.e.

$$\sigma_{\xi\xi\infty} = P\zeta^{2r},\ r = 0, 1, \ldots;\ \sigma_{ij\infty} = 0,\quad i = j \neq \xi \tag{1}$$

We write the state of stress of the plate in the form of a superposition of the basic stress $(\sigma_{ij}^\circ)$ appearing in a solid plate under the action of the load indicated, and the perturbed stress $(\sigma_{ij}^*)$ appearing in the plate with a cavity, the side surface of which is acted upon by the following forces:

$$\sigma_{rl/\Omega}^* = -\overset{\circ}{\sigma}_{rl/\Omega}\quad (l = r, \theta, \zeta;\ \sigma_{rl\infty}^* = 0) \tag{2}$$

The basic state of stress has the form /3/

$$\overset{\circ}{\sigma}_{rr} = \frac{P}{2}(1 + \cos 2\theta)\zeta^{2r},\quad \overset{\circ}{\sigma}_{r\theta} = -\frac{P}{2}\zeta^{2r}\sin 2\theta,\quad \overset{\circ}{\sigma}_{\theta\theta} = \frac{P}{2}(1 - \cos 2\theta)\zeta^{2r},\quad \overset{\circ}{\sigma}_{r\zeta} = \overset{\circ}{\sigma}_{\theta\zeta} = \overset{\circ}{\sigma}_{\zeta\zeta} = 0 \tag{3}$$

We obtain for the perturbed state of stress a boundary value problem of the theory of elasticity with boundary conditions (2), in which the right-hand sides are given by the relations (3). Solution of this problem reduces to finding the Lur'e–Lekhnitskii functions $F$, $\Phi_k, \Psi_p$ from the boundary conditions at the side surface /2/

$$\varphi(\sigma) + \sigma\overline{\varphi'(\sigma)} + \overline{\Psi(\sigma)} + {}^1/_2\Lambda_{1\Omega}(\Phi_0, \Psi_p) = \frac{1}{2}f_{1,0}(\sigma),\quad \left(\frac{4\lambda}{m\pi}\right)^2\mu_s\overline{\varphi''(\sigma)} - \Lambda_{1\Omega}(\Phi_m, \Psi_p) = f_{1,m};\quad \Lambda_{2\Omega}(\Phi_m, \Psi_p) = 0 \tag{4}$$

where

$$F = \operatorname{Re}[\bar z\varphi(z) + \chi(z)],\quad \Psi(z) = d_z\chi,\quad \nabla^2\nabla^2F = 0,\ \nabla^2\Phi_k = (\delta_k^*)^2\Phi_k,\quad \nabla^2\Psi_p = (\gamma_p^*)^2\Psi_p \tag{5}$$

We write the general solution of the system (3) satisfying the conditions of the boundedness of the stresses at infinity, in the form

$$\varphi(z) = a_1/z,\quad \Psi_p(r, \theta) = c_{0p}^*k_0(\gamma_p^*r) + C_{2p}^*k_2(\gamma_p^*r)\cos 2\theta,\quad \Psi(z) = \frac{b_1}{z} + \frac{b_3}{z^3},\quad \Phi_k(r, \theta) = b_{2k}^*k_2(\delta_k^*r)\sin 2\theta \tag{6}$$

Substituting (6) into the boundary conditions (4) we obtain an infinite system of linear algebraic equations for the arbitrary constants $a_1, b_1, b_3, c_{np} = c_{np}^*/k_n(\gamma_p^*)$, $b_{2k} = b_{2k}^*/k_2(\delta_k^*)$. We write it in its final form as follows:

$$\operatorname{Re}\sum_p [l_{mp} - n_{mp}P_0^-(\gamma_p^*)]c_{0p} = -\frac{P}{2}E_m^r,\quad \operatorname{Re}\sum_p r_{mp}P_0^-(\gamma_p^*)c_{0p} = 0\quad (p, m = 1, 2, \ldots, \infty) \tag{7}$$

$$b_1 = -\frac{P}{2}\left\{E_0^r + \operatorname{Re}\sum_p [l_{0p} - n_{0p}P_0^-(\gamma_p^*)]c_{0p}\right\},\ 3b_3 - 4a_1 + \operatorname{Re}\sum_p [l_{0p} + n_{0p}N_2^-(\gamma_p^*)]c_{2p} = -\frac{P}{4}E_0^r \tag{8}$$

$$3b_3 - 2a_1 - \operatorname{Re}\sum_p n_{0p}M_2^-(\gamma_p^*)c_{2p} = \frac{P}{4}E_0^r,\quad \frac{96\lambda^2\mu_s}{(m\pi)^2}a_1 - (-1)^m M_2^-(\delta_m^*)b_{2m} - \sum_p [l_{mp} + n_{mp}N_2^-(\gamma_p^*)]c_{2p} = \frac{P}{2}E_m^r$$

$$\frac{96\lambda^2\mu_s}{(m\pi)^2}a_1 + (-1)^m\left[\frac{1}{2}(\delta_m^*)^2 + N_2^-(\delta_m^*)\right]b_{2m} + \sum_n n_{mp}M_2^-(\gamma_p^*)c_{2p} = \frac{P}{2}E_m^r$$

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$$\sum_p r_{mp} P_2^- (\gamma_p^*) c_{2p} - \frac{(-1)^m \delta_m}{\lambda s_0} b_{2m} = 0$$

where we use the notation adopted in the monograph /2/.

Having determined from (7), (8), the constants introduced, we find the stresses in the plate using the formulas

$$\sigma_{ij} = \sigma_{ij}^* + \sigma_{ij}^\circ \tag{9}$$

where the quantities accompanied by an asterisk have the form

$$\frac{1}{P}\sigma_{\theta\theta}^* = \frac{b_1}{r_2} + \cos 2\theta \left\{ \frac{3b_3}{r^4} - \left[ \frac{4}{r^2} - \frac{24\lambda^2\mu_8}{r^4}\left(\frac{1}{3} - \zeta^2\right)\right] a_1 + \sum_k P_k(\zeta) M_2^-(\delta_k^* r) b_{2k}^* k_2(\delta_k^* r) + \right. \tag{10}$$

$$\left. \sum_p [l_p(\zeta) + n_p(\zeta) N_2^-(\gamma_p^* r)] c_{2p}^* k_2(\gamma_p^* r) \right\}$$

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$$\frac{1}{P}\sigma_{\zeta\zeta}^* = 2\,\mathrm{Re} \sum_p t_p(\zeta) c_{2p}^* k_2(\gamma_p^* r) \cos 2\theta \quad (k, p = \overline{1, \infty})$$

In investigating the concentration of the stresses appearing in the plate in question, we use the values of the stresses on $\Omega$. In particular, for the normal stresses we have

$$\frac{1}{P}\sigma_{\theta\theta}|_\Omega = A(\lambda,\zeta) - B(\lambda,\zeta)\cos 2\theta, \; \frac{1}{P}\sigma_{\zeta\zeta}|_\Omega = c(\lambda,\zeta)\cos 2\theta, \; A(\lambda,\zeta) = \frac{1}{2}\zeta^{2r} - b_1, \; c(\lambda,\zeta) = 2\,\mathrm{Re}\sum_p t_p(\zeta) c_{2p} \tag{11}$$

$$B(\lambda,\zeta) = \frac{1}{2}\zeta^{2r} + 24\lambda^2\mu_8\left(\frac{1}{3} - \zeta^2\right) a_1 + \sum_k P_k(\zeta) M_2^-(\delta_k^*) b_{2k} + 3b_3 - 2\,\mathrm{Re}\sum_p [s_p(\zeta) - n_p(\zeta) N_2^-(\gamma_p^*)] c_{2p}$$

Expressions (11) imply that the greatest normal stresses appear in the cross-section $\theta = \pi/2$. Let $r = 0$. Then the system (7) yields $c_{0p} = 0$, $b_1 = -P/2$. Table 1 shows the maximum values of the normal stresses for this case, for cadmium (Cd) and zinc (Zn) plates, for various values of the relative thickness $\lambda$. Comparison with the corresponding values for the isotropic plates /2/ shows that the character of the stress distribution across the thickness and along the circumference is the same for the transtropic and the isotropic plates. The greatest discrepancy in the values of the peripheral stresses is observed near the plane edges, and it can reach 20%. The stresses $\sigma_{\zeta\zeta}$ appearing in the transtropic plates are almost twice as large as those in the isotropic plates. Consequently, the stress field in transtropic plates has a more pronounced three-dimensional character than that in the isotropic plates. It follows that if the anisotropy of the material is to be taken into account, we must use the three-dimensional solution when computing the stress concentration.

Table 1

| $\zeta$ | $\sigma_{\theta\theta}\|_\Omega$ | | | $\sigma_{\zeta\zeta}\|_\Omega$ | | |
|---|---|---|---|---|---|---|
| | $\lambda = 1$ | | $\lambda = 4$ | $\lambda = 1$ | | $\lambda = 4$ |
| | Cd | Zn | Cd | Cd | Zn | Cd |
| 0.2 | 3.343 | 3.516 | 3.084 | 0.418 | 0.538 | 0.538 |
| 0.4 | 3.295 | 3.415 | 3.108 | 0.368 | 0.455 | 0.537 |
| 0.6 | 3.192 | 3.250 | 3.150 | 0.277 | 0.341 | 0.522 |
| 0.8 | 2.967 | 2.947 | 3.164 | 0.142 | 0.189 | 0.435 |
| 1.0 | 2.385 | 2.247 | 2.167 | $0.1 \cdot 10^{-6}$ | $0.8 \cdot 10^{-6}$ | $0.2 \cdot 10^{-6}$ |

If the plate is subjected to a tensile force $Q$ acting in the direction of the axis $0\eta$ ($\sigma_{\eta\eta\infty} = Q$) then the solution of the problem of the state of stress in such a plate is given by the formulas (10), (11) where $P$ is replaced by $Q$ and $\theta$ by $\theta + \pi/2$. In particular, for the stresses near the surface $\Omega$ we have

$$\frac{1}{Q}\sigma_{\theta\theta}|_\Omega = A(\lambda,\zeta) + B(\lambda,\zeta)\cos 2\theta, \quad \frac{1}{Q}\sigma_{\zeta\zeta}|_\Omega = -c(\lambda,\zeta)\cos 2\theta$$

Superimposing the solutions in question corresponding to the loads $\sigma_{\xi\xi\infty} = P\zeta^{2r}$ and $\sigma_{\eta\eta\infty} = Q\zeta^{2r}$, we obtain a solution for the case when an unbounded plate is subjected to the tensile forces $P\zeta^{2r}$ and $Q\zeta^{2r}$ acting, respectively, along and across the $0\xi$-axis. In this case we have

$$\sigma_{\theta\theta}|_{\Omega} = (P + Q)A(\lambda, \zeta) - (P - Q)B(\lambda, \zeta)\cos 2\theta, \quad \sigma_{zz}|_{\Omega} = (P - Q)C(\lambda, \zeta)\cos 2\theta$$

In particular, when the plate is under equilateral tension, we have

$$\sigma_{\theta\theta}|_{\Omega} = 2PA(\lambda, \zeta)$$

The system (8) yields in this case $a_1 = b_3 = c_{2p} = b_{3k} = 0$. An analogous problem was solved in /4/.

Table 2 gives the value for the stresses $(\sigma_{ij}^*)$ of the perturbed state for different quantities of the eigenfunctions left in the solution. The data are obtained for a Cd plate acted upon by the load $\sigma_{rr}^*|_{\Omega} = -P\zeta^2$ and $\lambda = 1$. The Table shows how well the boundary conditions are satisfied (the last column corresponds to the stresses given in advance), and how rapidly the approximate solutions converge in the Bubnov—Galerkin method. Already for $m = 8$ the stress values can be assumed exact. Even at $m = 3$ the boundary conditions for the plates with $\lambda = 1$ and $\lambda = 4$ are satisfied with an error not exceeding 1.5% of the loads given. When $\lambda = 0.1$, $m = 3$ yields results which are practically exact. The convergence becomes less rapid with the increasing value of the index $r$ of variation of the load. A similar pattern is observed when the problems are solved for a Zn plate. The character of variation in the perturbed state of stress on moving away from the edge into the middle of the plate is of considerable interest.

Table 2

| $\sigma_{ij}^*$ | $\zeta$ | $m = 3$ | 5 | 10 | 20 | given stresses |
|---|---|---|---|---|---|---|
| $\sigma_{rr}^*\|_{\Omega}$ | 0.2 | −0.0389 | −0.0396 | −0.03999 | −0.040006 | −0.04 |
| | 0.6 | −0.3623 | −0.3596 | −0.35997 | −0.360003 | −0.36 |
| | 0.8 | −0.6356 | −0.6401 | −0.63993 | −0.639997 | −0.64 |
| | 1 | −1.0139 | −1.0065 | −1.00132 | −1.000014 | −1 |
| $10^{-5}\cdot\sigma_{r\zeta}^*\|_{\Omega}$ | 0.2 | 80 | 10 | 1 | 0.2 | 0 |
| | 0.6 | 90 | −10 | 4 | 1.2 | 0 |
| | 0.8 | −310 | 30 | 8 | 2.9 | 0 |
| $\sigma_{\theta\theta}^*\|_{\Omega}$ | 0.2 | 0.3172 | 0.3178 | 0.31789 | 0.31786 | |
| | 0.4 | 0.3645 | 0.3647 | 0.36489 | 0.36485 | |
| | 0.6 | 0.4380 | 0.4379 | 0.43803 | 0.43799 | |
| | 0.8 | 0.5295 | 0.5315 | 0.53182 | 0.53177 | |
| | 1 | 0.6548 | 0.6549 | 0.65491 | 0.65490 | |

Table 3 gives the corresponding data for a Zn plate under the load $\sigma_{rr}^*|_{\Omega} = -P\zeta^2$ (the stresses $\sigma_{rr}^*$ are given in the numerator, and $\sigma_{\theta\theta}^*$ in the denominator). Analysis of the data given in the Table uncovers a distinctive feature in the stress distribution in the plate at various values of the thickness, which implies that the peripheral stresses concentrate near the surface and decay sufficiently rapidly on moving towards the inside of the plate.

Table 3

| $\lambda$ | $r$ | $\zeta = 0$ | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|---|
| 0.5 | 1 | $\frac{0}{0.373}$ | $\frac{-0.040}{0.383}$ | $\frac{-0.160}{0.413}$ | $\frac{-0.360}{0.462}$ | $\frac{-0.640}{0.537}$ | $\frac{-1}{0.662}$ |
| | 1.6 | $\frac{-0.061}{0.074}$ | $\frac{0.071}{0.080}$ | $\frac{-0.098}{0.095}$ | $\frac{-0.138}{0.117}$ | $\frac{-0.180}{0.139}$ | $\frac{-0.218}{0.153}$ |
| | 2.4 | $\frac{-0.062}{-0.056}$ | $\frac{-0.061}{0.056}$ | $\frac{-0.060}{0.056}$ | $\frac{-0.057}{0.056}$ | $\frac{-0.054}{0.056}$ | $\frac{-0.050}{-0.055}$ |
| 1 | 1 | $\frac{0}{0.259}$ | $\frac{-0.040}{0.280}$ | $\frac{-0.160}{0.342}$ | $\frac{-0.360}{0.443}$ | $\frac{-0.640}{0.582}$ | $\frac{-1}{0.768}$ |
| | 1.8 | $\frac{0.016}{0.063}$ | $\frac{-0.002}{0.071}$ | $\frac{-0.055}{0.095}$ | $\frac{-0.139}{0.133}$ | $\frac{-0.243}{0.179}$ | $\frac{-0.348}{0.225}$ |
| | 3.2 | $\frac{-0.032}{0.026}$ | $\frac{-0.033}{0.027}$ | $\frac{-0.033}{0.028}$ | $\frac{-0.033}{0.029}$ | $\frac{-0.032}{0.030}$ | $\frac{-0.029}{0.030}$ |
| 4 | 1 | $\frac{0}{0.061}$ | $\frac{-0.040}{0.098}$ | $\frac{-0.160}{0.209}$ | $\frac{-0.360}{0.391}$ | $\frac{-0.640}{0.639}$ | $\frac{-1}{0.942}$ |
| | 1.6 | $\frac{0.015}{0.028}$ | $\frac{-0.001}{0.042}$ | $\frac{-0.051}{0.083}$ | $\frac{-0.013}{0.150}$ | $\frac{-0.252}{0.242}$ | $\frac{-0.399}{0.350}$ |
| | 4 | $\frac{0.007}{0.004}$ | $\frac{0.003}{0.006}$ | $\frac{0.006}{0.011}$ | $\frac{-0.021}{0.020}$ | $\frac{-0.040}{0.030}$ | $\frac{-0.058}{0.040}$ |

It should be noted that in the case when the external loads vary across the thickness, the stress distribution pattern in an isotropic plate is basically different from that in a transtropic plate. Thus, if in the isotropic plate the largest compressive stresses act on the middle plane of the plate, then in the Cd and Zn plates the maximum is reached at the edge of the plate. From this it follows that inclusion of the anisotropy leads to a completely new result.

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